

Textures and Newtonian Gravity

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Newtonian theory is used to study the gravitational effects of a texture, in particular the formation of massive structures.

PACS numbers: 04.20.-q, 11.17.+y, 98.80.Bp

Topological defects: domain walls, strings and monopoles have been recently the focus of much attention in cosmology [1]–[3]. The symmetry breaking of non-Abelian groups always leads to the formation of textures, a cosmic defect that can be asymptotically represented as a space-time point, an event [4,5]. In this case the homotopy theory tells us that the vacuum manifold topology is not trivial, $\pi_3(M) \approx Z$.

Textures are global defects that can give rise to seeds of cosmic background anisotropies [6,7]. In principle, we may observe its effect by looking to the microwave spectra fluctuations, like the ones obtained by the satellite COBE [8]. The aim of this note is to study the mass accretion by a texture in the simplest way. We shall see that the use of Newtonian gravity, the zeroth order approximation to general relativity, makes possible to visualize the formation of massive structures in a very simple way.

The Lagrangian density associated to textures with $SU(2)$ symmetry can be written as,

$$\mathcal{L} = \frac{1}{2} \partial^\mu \Phi^{\dagger a} \partial_\mu \Phi^a - \frac{\lambda}{4} (\Phi^a \Phi^{\dagger a} - \eta^2)^2. \quad (1)$$

Since, we are interested only in the scalar sector of the theory, instead of $SU(2)$ we shall require $SO(4)$ symmetry. Then, the scalar fields may be represented as a real quadruplet. To break the symmetry we introduce in the action the constrain $\Phi^a \Phi^a = \eta^2$ via a Lagrangian multiplier α . Hence, the symmetry will be broken from $SO(4)$ to $SO(3)$. This is possible because for textures the third homotopy group of the vacuum manifold must be non trivial and $\pi_3(SO(4)/SO(3)) \approx \pi_3(SU(2)/1) \approx Z$.

The action in Minkowski spacetime (M_4),

$$S = \int \frac{1}{2} \partial^\mu \Phi^a \partial_\mu \Phi^a - \frac{\lambda}{4} (\Phi^a \Phi^a - \eta^2)^2 + \alpha (\Phi^a \Phi^a - \eta^2) d^4x, \quad (2)$$

yields the texture field equations,

$$\square \Phi^a = - \frac{\nabla_\mu \Phi^b \nabla^\mu \Phi^b}{\eta^2} \Phi^a. \quad (3)$$

The constrain $\Phi^a \Phi^a = \eta^2$ tells us that the texture can be regarded as a 3-sphere located in a four dimensional field space. Therefore, the most natural parametrization [4] of Φ^a is

$$\begin{aligned} \Phi^{(0)} &= \eta \cos \chi(r, x^0) \\ \Phi^{(i)} &= \eta [\sin \chi(r, x^0) \cos \varphi \sin \theta \sin \chi(r, x^0) \sin \varphi \sin \theta, \\ &\quad \sin \chi(r, x^0) \cos \theta], \end{aligned} \quad (4)$$

where, $x^0 = ct$. The motion equations reduces to a single one for the function $\chi(r, x^0)$,

$$\frac{\partial^2 \chi}{\partial x^{02}} - \frac{2}{r} \frac{\partial \chi}{\partial r} - \frac{\partial^2 \chi}{\partial r^2} = \frac{\sin 2\chi}{r^2}. \quad (5)$$

The study of the Lie symmetries of this equations gives us the similarity variables, $u = r/x^0$, and r . The first one reduces (5) to

$$\chi'' + \frac{2}{u} \chi' = \frac{\sin 2\chi}{u^2(1-u^2)}. \quad (6)$$

The prime indicates derivation with respect to the similarity variable $u = r/x^0$.

It is a simple exercise to prove that the geodesic equations for particles moving with non-relativistic speed ($v \ll c$) in the presence of a weak gravitational field $g_{\mu\nu}(x^\lambda) = \eta_{\mu\nu} + h_{\mu\nu}(x^\lambda)$, ($|h_{\mu\nu}|^2 \ll 1$), can be cast in the the Newtonian form

$$\frac{d^2 \mathbf{r}}{dt^2} = -\frac{1}{2} c^2 \nabla h_{00}. \quad (7)$$

Thus, in this approximation the component h_{00} plays, essentially, the role of the Newtonian potential, and the other components play no role at all. Therefore, we shall consider that the weak gravitational field associated with the texture can be described by the spherically symmetric metric,

$$ds^2 = [1 + h_{00}(t, r)] c^2 dt^2 - dr^2 - r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (8)$$

By solving the eigenvalue equation for the Ricci tensor [$\det(R^\mu_\nu - \lambda \delta^\mu_\nu) = 0$] derived from (8), we get for the eigenvalue associated to the timelike eigenvector,

$$\lambda_0^R = (h_{00,rr} + 2h_{00,r}/r)/2, \quad (9)$$

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that in this case is just R_t^t .

Recalling that the eigenvalues values of matrices N and M such that $M = N + \alpha I$ are related by $\lambda_M = \lambda_N + \alpha$, we have that the Einstein equations $R_\nu^\mu = \frac{8\pi G}{c^2}(T_\nu^\mu - \frac{1}{2}T\delta_\nu^\mu)$ give us for the eigenvalue related to the timelike eigenvector

$$\lambda_0^R = \frac{8\pi G}{c^2}(\lambda_0^T - \frac{1}{2}T). \quad (10)$$

The trace of the energy-momentum tensor (EMT) can be always written as the sum of its eigenvalues. The eigenvalue related to the timelike eigenvalue is defined as the material density times the square of the light velocity ρc^2 , and the eigenvalues related to the spacelike eigenvectors are denoted as: $-p_1, -p_2$, and $-p_3$ [10]. The quantities p_i when positive (negative) are the principal pressures (tensions). From (10), (9) and the definition $\psi \equiv c^2 h_{00}/2$ we find

$$\nabla^2 \psi = 4\pi G \rho_N \quad (11)$$

where ρ_N is the associated Newtonian density,

$$\rho_N = \rho + (p_1 + p_2 + p_3)/c^2. \quad (12)$$

From (7) we find

$$\frac{d^2 \mathbf{r}}{dt^2} = -\nabla \psi. \quad (13)$$

The concept of associated Newtonian density of a given distribution of matter was first developed by Tolman [11] for quasi-static metrics, lately MacCrea and Milne [12] found the Friedmann-Lemaître equation for the expansion rate of the universe radius using Newtonian considerations [13]. For applications of this concept to cosmic strings and domain walls see Refs. [2].

We shall begin our considerations about structure formation by studying the texture energy-momentum tensor. The metric and the canonical EMT are equal in this case. This tensor can be computed for the generic solution of equation (6), $\chi(r/t)$ (from now on we shall use units such that $c = 1$). We find,

$$(\hat{T}_\nu^\mu) = \frac{\eta^2(\chi')^2}{2t^2} \times \begin{pmatrix} 1 + u^2 + F^2 & 2u^2 & 0 & 0 \\ -2u^2 & -1 - u^2 + F^2 & 0 & 0 \\ 0 & 0 & 1 - u^2 & 0 \\ 0 & 0 & 0 & 1 - u^2 \end{pmatrix}, \quad (14)$$

where $F = 2 \sin^2 \chi / (u\chi')^2$. By solving the eigenvalue problem associated with (14) we find its diagonal form,

$$(T_\nu^\mu) = \frac{\eta^2(\chi')^2}{2t^2} \times \begin{pmatrix} |1 - u^2| + F^2 & 0 & 0 & 0 \\ 0 & -|1 - u^2| + F^2 & 0 & 0 \\ 0 & 0 & 1 - u^2 & 0 \\ 0 & 0 & 0 & 1 - u^2 \end{pmatrix}. \quad (15)$$

The EMT (15) indicates that the associated gravitational density ρ_N vanishes when $r < |t|$ and it is positive when $r > |t|$. The relation (12) tells us that not only the mass density gravitates, but also the associated pressures contribute positively to this mass density and the tensions negatively. This simple observation allows us to foresee some aspects of the microwave distortions produced by a texture.

Let us consider a texture localized at the origin of M_4 , $r = t = 0$ and its corresponding light cone (Fig.1).

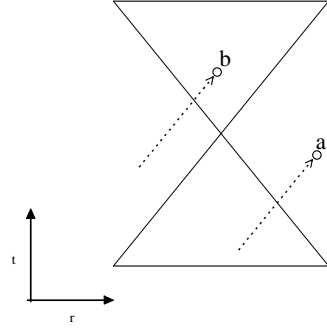


FIG. 1. The photon a is red-shifted by the presence of the texture at $t = r = 0$ whilst the photon b is blue-shifted.

Photons entering the cone are attracted towards the center, their energy increase (they are blue-shifted). The photons leaving the cone are also attracted, but their energy decrease (they are red-shifted). This result was obtained by Turok and Spergel [7] using the Einstein equations in the weak field approximation with the EMT associated to the special solution of (6),

$$\chi(r/t) = \begin{cases} 2 \arctan(-r/t), & t < 0 \\ 2 \arctan(r/t) + \pi & 0 < r < t \\ 2 \arctan(t/r) + \pi & 0 < t < r \end{cases}, \quad (16)$$

which represents an unwinding texture that changes its topological charge at $t = 0$.

Now we shall study the effect of the texture on matter using Newtonian gravitation, that is the zeroth order approximation to general relativity Eqs. (11)-(13). By numerically solving the Barriola and Vachaspati [14] equations for a self-gravitating texture we find that the weak field condition is fulfilled [15].

From (16), (12), and (15) we find

$$\rho_N = \begin{cases} 0, & r \leq |t| \\ \frac{8\eta^2(r^2 - t^2)}{(r^2 + t^2)^2}, & r > |t| \end{cases}. \quad (17)$$

It is important to emphasize that the singularity for this solution is exactly at the point $(t = 0, r = 0)$. By solving

(11) with (17) as a source we get the Newtonian gravity of the texture $\mathbf{G} = -\nabla\psi$,

$$G_r = 8\pi G\eta^2 \left[\frac{[2r - 3t + \pi t - 4t \arctan(r/t)]}{2r^2} + \frac{t^2}{r(r^2 + t^2)} \right], \quad (18)$$

in the region $|t| < r$ and zero elsewhere. We can use (18) to make simple simulations to describe the gravitational action of a texture in the following way: Firstly we numerically solve the Newtonian motion equation for a particle subject to the central force per unit of mass G_r . The initial conditions are the ones for an homogeneous medium. When the time variable t reaches the value of the i^{th} particle radial coordinate, r_i , we stop the integration. At $t = r_i$ the density ρ_N and the associated gravitational potential are zero. Then, at this instant the particle starts to move with constant velocity. This velocity is taken to be the same computed previously. In Fig. 2 we show the initial position of a distribution of particles. Fig. 3 presents a mass concentration due to the attractive character of the gravitational force of the texture.

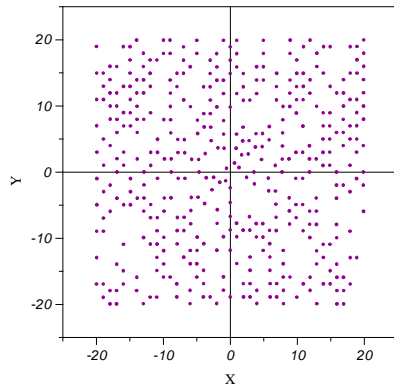


FIG. 2. Initial configuration.

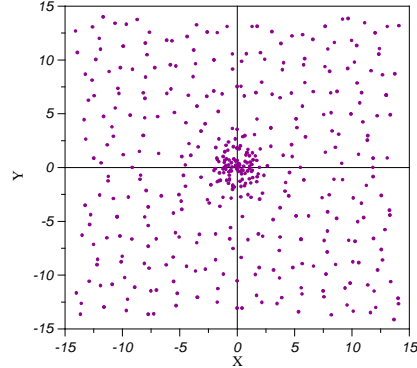


FIG. 3. Maximum density configuration.

In Fig. 4 we see the concentration of particles being diluted for t larger than all t_i . The matter is less concentrated than the homogeneous initial form.

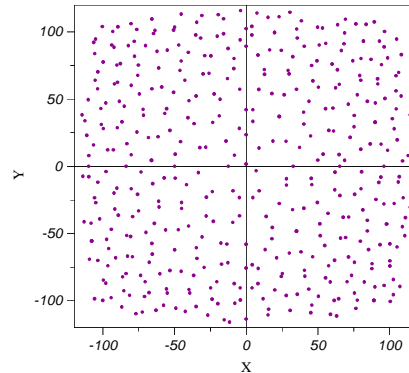


FIG. 4. Mass distribution after some time. This configuration expands forever.

In the simulations the texture parameter η has been greatly magnified, we used $\eta = 10^{19} \text{ GeV}$, a thousand times larger than the usual one. The configurations shown in the graphics do not depend on the space time units. The time variation from Fig. 2 to Fig. 4 is approximately 10^{-6} seconds for the length unit meter. Also, in order to have a meaningful Newtonian picture, in the simulations the velocity of the test particles at most reached a few percents of the speed of light. Some authors simulate the radiation density fluctuation produced by a tex-

ture using an approximate solution for small $u = r/t$ [16,17]. This region is exactly where the Newtonian density does not vanish. Therefore we can use our result to estimate the matter density anisotropy produced by a texture calculating the mass concentration at the instant represented in the Fig.3 and compare it with the initial one, Fig. 2.

The η value is the order of 10^{16}GeV [6], using the Planck mass we get

$$\frac{\delta\rho_m}{\rho_m} \simeq 10^{-3}. \quad (19)$$

Phillips [18] found that this kind of texture is not relevant to seed the microwave background density fluctuation because this event is much rarer than the less energetic ones. This fact might reduce the importance of this approach.

However the main aspect to be considered is that events like the Turok's texture are more energetic than others because of the potential barrier of the unwinding process [19]. We can speculate that the texture analyzed here can form an isolated very massive structure, e.g. The Great-Attractor [20].

In summary using a very simple Newtonian simulation one can obtain a good qualitative picture of the structure formation due to textures.

The authors thank FAPESP, CNPq and CAPES for financial support.

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